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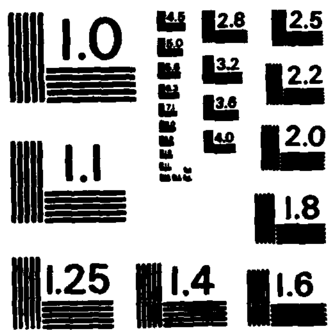

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Construction of Exponential Martingales

for Counting Processes

by

Walter A. Rosenkrantz[1]

Department of Mathematics and Statistics
University of Massachusetts
Amherst, MA 01003

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Abstract

Abstract

Let $N(t)$ be a counting process with continuous compensator $A(t)$ and $f(t)$ a bounded predictable process. If $E(\exp(2|f|N(t))) < \infty$ and $E(\exp(2(1 + \exp|f|)A(t))) < \infty$ then it is shown that $z(t) = \exp\{-\int_0^t f(u)dN(u) - \int_0^t [\exp(-f(u)) - 1]dA(u)\}$ is a martingale. This is a partial extension of a theorem of Kabanov, Liptser, Shiryaev (1980) who assumed $A(t) \leq c$ but did not assume $A(t)$ is continuous.

1. Introduction

If $p(t)$ is a standard Poisson process of unit intensity with $P(p(t) = j) = \exp(-t)t^j/j!$, $j = 0,1,\ldots$ then it is easy to see that

$$(1.1)\qquad z(t) = \exp\{-\lambda p(t) - (e^{-\lambda}-1)t\} \text{ is a martingale for every } \lambda \in R.$$

Formula (1) suggests that if f is bounded and predictable with respect to the filtration $F(t) = \sigma(p(s), 0 \leq s \leq t)$ then

$$(1.2)\qquad z(t) = \exp\{-\int_0^t f(u)dp(u) - \int_0^t [\exp(-f(u)) - 1]du\}$$

is a martingale also. Note that by putting $f(u) \equiv \lambda$ in (1.2) we obtain (1.1). More generally Kabanov-Liptser-Shiryaev (1980) (henceforth abbreviated to K-L-S) have proved the following theorem.

THEOREM 1: Let $N(t)$ denote a counting process with continuous compensator $A(t)$ satisfying the condition $A(t,\omega) \leq c$ and let $f(t,\omega)$ denote a bounded predictable process with respect to the filtration $F(t) = \sigma(N(s), 0 \leq s \leq t)$. Then

$$(1.3)\qquad z(t) = \exp\{-\int_0^t f(u)dN(u) - \int_0^t [\exp(-f(u)) - 1]dA(u)\}$$

is a martingale.

Remarks: (i) K-L-S use the martingale $z(t)$ to give a very nice proof of a Poisson limit theorem for point processes due to T. Brown (1978).

(ii) If we recall that the compensator of the Poisson process $A(t) = t$ then we see at once that the condition $A(t) \leq c$ is too restrictive since it excludes the Poisson process! These remarks suggest that a more natural condition to impose on $A(t)$ in order for the process $z(t)$ defined by (1.3) above to be a martingale is the following one:

$$(1.4) \qquad E(\exp(cA(t))) < \infty, \quad E(\exp(dN(t)) < \infty$$

for non-negative constants c and d which may depend on $|f|$.

It is the purpose of this paper to give a statement and proof of just such an extension to Theorem 1.

THEOREM 2: Let $N(t)$ denote a counting process with continuous compensator $A(t)$ and let $f(t,\omega)$ denote a bounded predictable process.

(i) If $A(t)$ satisfies condition (1.4) with $c = 2(1 + \exp(|f|))$ and $d = 2|f|$ then the process $z(t)$ (defined at (1.3)) is a martingale.

(ii) If in addition $f(t,\omega) \geq 0$ and $A(t)$ satisfies condition (1.4) with $c = 1$ and $d = 0$ then $z(t)$ is a martingale.

When the hypothesis that $A(t)$ be continuous is dropped K-L-S (1980) have shown that

$$(1.8) \qquad z(t) = \exp\{-\int_0^t f(u)dN(u) - \int_0^t [\exp(-f(u)) - 1]du - \sum_{s\leq t} \Phi(\exp(f(s)) - 1)\Delta A(s)\}$$

is a martingale provided $A(t) \leq c$ where $\Phi(x) = \ell n(1 + x) - x$. We conjecture that (1.6) remains true under the less restrictive condition (1.4); the proof of this result however has so far escaped us.

Outline of the Proof: We first prove Theorem 2 in the special case where $N(t) = p(t)$. The general case is then reduced to this one via a random time change. A similar method is used by Ikeda-Watanabe in their proof of a Theorem of Novikov's cf. IKEDA-WATANABE (1981) Theorem 5.3 pp 142-144.

NOTATION: Whenever convenient we will drop the ω and write $f(t)$ for $f(t,\omega)$, $A(t)$ for $A(t,\omega)$ etc.

2. Proof of Theorem 2:

Recall the setting of the introduction: $p(t)$ is a standard Poisson process of unit intensity and $F(t) = \sigma(p(s); 0 \leq s \leq t)$.

LEMMA 1: If $X(\omega)$ is $F(s)$ measurable and bounded then

$$E(\exp(-X(\omega)[p(t) - p(s)]|F(s)) = \exp([t - s](\exp(-X(\omega)) - 1)).$$

This is a consequence of the following lemma, the proof of which is left to the reader.

LEMMA 2: Let $h(x,y)$ be a bounded Borel measurable function and suppose $X(\omega)$ and $Y(\omega)$ are random variables such that $X(\omega)$ is G measurable. Then

$$E(h(X,Y)|G) = g(X(\omega),\omega) \quad \text{where}$$
$$g(x,\omega) = E(h(x,Y)|G).$$

LEMMA 3: Let $f(t,\omega)$ be a bounded $F(t)$ adapted process with left continuous paths (and hence predictable). Then

$$\exp\left(-\int_0^t f(u)dp(u) - \int_0^t [\exp(-f(u)) - 1]du\right) \text{ is a martingale.}$$

Proof: Assume f is a simple function i.e.

$$f(u,\omega) = \sum_{i=0}^{n-1} f(t_i,\omega)I_{(t_i,t_{i+1}]}(u) \quad \text{where} \tag{2.1}$$

$0 = t_0 < t_1 < \dots < t_n$. It suffices to show that

$$E\left(\exp\left(-\int_s^t f(u)dp(u) - \int_s^t [\exp(-f(u)) - 1]du\right) \Big| F(s)\right) = 1 \quad 0 \le s < t. \tag{2.2}$$

Assume $t_i \le s < t \le t_{i+1}$ so $\int_s^t f(u)dp(u) = f(t_i,\omega)(p(t) - p(s))$ and $\int_s^t [\exp(-f(u)) - 1]du = (t - s)(\exp(-f(t_i,\omega)) - 1)$ which is $F(s)$ measurable. Consequently by Lemma 1

$$E\left(\exp\left(-\int_s^t f(u)dp(u)\right) \Big| F(s)\right) = E(\exp(-f(t_i,\omega)(p(t) - p(s)) | F(s))$$

$$= \exp\{(t - s)[\exp(-f(t_i,\omega)) - 1]\} \quad \text{which yields (2.2).}$$

If $s < t_{i+1} < t$ then we can reduce it to the case just considered by successively conditioning on $F(t_{i+1})$ and then $F(s)$ etc.

For the next step we invoke Lemma 5.3 on p. 175 of Liptser-Shiryaev V.1 (1977) which asserts that sample functions of the form (2.1) are dense in the class of predictible functions satisfying the condition

$$E\left(\int_0^t (f(u,\omega))^2 dA(u)\right) < \infty. \tag{2.3}$$

Here density is of course understood to be with respect to the norm
$E(\int_0^t (f(u,\omega) - g(u,\omega))^2 dA(u))^{1/2}$.

Bring in the square integrable martingale $M(t) = p(t) - t$ and recall that the compensator of $(M(t))^2$ is t. Let $f_n(t,\omega)$ denote a sequence of simple functions of the form (2.1) satisfying the conditions $|f_n| \leq |f|$ and

$$(2.4) \qquad \lim_{n\to\infty} E(\int_0^t (f_n(u,\omega) - f(u,\omega))^2 du) = 0, \quad \text{i.e. set } A(u) = u \text{ in } (2.3);$$

It then follows that

$$(2.5) \qquad \lim_{n\to\infty} E(|\int_0^t f_n(u,\omega)dM(u) - \int_0^t f(u,\omega)dM(u)|^2) = 0.$$

Applying Schwarz's inequality and (2.4) we see that

$$(2.6) \qquad \lim_{n\to\infty} E(\int_0^t |f_n(u) - f(u)|du) \leq \sqrt{t} \lim_{n\to\infty} E(\int_0^t |f_n(u) - f(u)|^2 du) = 0.$$

In addition the condition $|f_n| \leq |f|$ implies that

$$|\int_0^t \exp(-f_n(u))du - \int_0^t \exp(-f(u))du| \leq K \int_0^t |f_n(u) - f(u)|du;$$

thus

$$(2.7) \qquad \lim_{n\to\infty} E(|\int_0^t \exp(-f_n(u))du - \int_0^t \exp(-f(u))du|) = 0 \quad .$$

Next we observe that $\int_0^t f_n(u)dp(u) + \int_0^t [\exp(-f_n(u)) - 1]du =$
$\int_0^t f_n(u)dM(u) + \int_0^t \exp(-f_n(u))du$ and that

(2.8) $$\left|\int_0^t f_n(u)dp(u) + \int_0^t [\exp(-f_n(u)) - 1]du\right| \le |f|p(t) + t(1 + \exp(|f|)) .$$

Consequently

(2.9) $$|\exp\{-\int_0^t f_n(u)dp(u) + \int_0^t [\exp(-f_n(u)) - 1]du\}| \le$$
$$\exp(|f|p(t) + t(1 + \exp|f|)$$

which is obviously an integrable function. It is clear we can now extract a subsequence $f_{n'}(u)$ such that

(2.10) $$\begin{cases} \text{(a)} \quad \lim_{n'\to\infty} \int_0^t f_{n'}(u)dM(u) = \int_0^t f(u)dM(u) \quad \text{a.s.} \\ \text{(b)} \quad \lim_{n\to\infty} \int_0^t \exp(-f_{n'}(u))du = \int_0^t \exp(-f(u))du \quad \text{a.s.} \end{cases}$$

On the other hand we've already shown for simple functions $f_{n'}$ that

(2.11) $$E(\exp\{-\int_s^t f_{n'}(u)dp(u) - \int_s^t \exp(-f_{n'}(u)) - 1)du\}|F(s)) = 1.$$

The bound (2.9) and the existence of the limits in (2.10) now permit us to pass to the limit in (2.11) and deduce that (2.2) remains valid for bounded predictable f. Q.E.D.

LEMMA 4: Let $N(t)$ be a counting process with a continuous strictly increasing compensator $A(t)$ satisfying condition (1.4) with $c = 2(1+\exp|f|)$ and $d = 2|f|$ (or $c = 1, d = 0$ if $f(t) \ge 0$). Then

(2.12) $$z(t) = \exp\{-\int_0^t f(u)dN(u) - \int_0^t [\exp(-f(u)) - 1]dA(u)\}$$

is a martingale.

Proof: Bring in the random time change $A^{-1}(t) = \inf\{u : A(u) > t\}$ and note that $A^{-1}(t)$ is also continuous and strictly increasing. It is easy to see that $N(A^{-1}(t))$ is again a counting process with compensator $A(A^{-1}(t)) = t$ and therefore $N(A^{-1}(t)) = p(t)$ is a Poisson process relative to the filtration $F'(t) = F(A^{-1}(t))$. Assume f is left continuous which implies that $f(A^{-1}(t))$ is predictable and therefore by Lemma 3

$$(2.13) \qquad \exp\{-\int_0^t f(A^{-1}(u))dN(A^{-1}(u)) - \int_0^t [\exp(-f(A^{-1}(u)) - 1]du\} = v(t)$$

is a martingale. Now $A(t)$ is a stopping time relative to the filtration $F'(t) = F(A^{-1}(t))$ and so Doob's optimal stopping theorem implies $v(t \wedge A(s))$ is also a martingale. Let us assume that $f(t) \geq 0$ which, combined with the fact that $N(A^{-1}(u))$ is monotone increasing, implies the inequality

$$(2.14) \qquad -\int_0^{t\wedge A(s)} f(A^{-1}(u)dN(A^{-1}(u)) - \int_0^{t\wedge A(s)}[\exp(-f(A^{-1}(u)) - 1]du \leq t \wedge A(s).$$

Consequently $0 \leq v(t \wedge A(s)) \leq \exp(t \wedge A(s)) \leq \exp(A(s))$. We may now apply the dominated convergence to conclude $\lim_{t\to\infty} v(t \wedge A(s)) = v(A(s))$ in L_1 and hence $v(A(s))$ itself is a martingale. Now

$$(2.15) \qquad v(A(s)) = \exp\{-\int_0^{A(s)} f(A^{-1}(u))dN(A^{-1}(u)) - \int_0^{A(s)}[\exp(-f(A^{-1}(u)) - 1)du\}$$

$$= \exp\{-\int_0^s f(u)dN(u) - \int_0^s [\exp(-f(u)) - 1]dA(u)\}$$

$$= z(s) \text{ is a martingale.}$$

We have thus proved (ii) of Theorem 2, at least in the case where f is continuous and $A(t)$ is strictly increasing. It is easy to extend this

result to simple functions of the form (2.1) by means of the following device: for each i construct a sequence of non-negative continuous functions $\phi_{k,i}(t)$, with compact support, such that $\lim_{k\to\infty} \phi_{k,i}(t) = I_{(t_i,t_{i+1}]}(t)$. Set $f_k(t) = \sum_{i=0}^{n-1} f(t_i,\omega)\phi_{k,i}(t)$ and note that we can arrange matters so that $f_k(t)$ is $F(t)$ adapted as well. Clearly $\lim_{k\to\infty} f_k(t) = f(t)$ in the sense of bounded pointwise convergence and from this it is easy to see that (ii) of Theorem 2 remains valid for non-negative simple functions of the form (2.1). The extension to arbitrary non-negative bounded predictable processes via the methods used in deriving (2.4)-(2.11) is left to the reader.

If we assume that f is bounded then inequality (2.14) is replaced by

$$(2.16) \qquad \left|\int_0^{t\wedge A(s)} f(A^{-1}(u))dN(A^{-1}(u)) + \int_0^{t\wedge A(s)} [\exp(-f(A^{-1}(u)) - 1]du\right| \leq$$

$$|f|p(A(s)) + (1 + \exp(|f|))A(s) = |f|N(s) + KA(s).$$

By Schwarz's inequality a sufficient condition for the integrability of $\exp(|f|N(s) + KA(s))$ is given by condition (1.4) with $c = 2K = 2(1 + \exp(|f|)$ and $d = 2|f|$. The proof of Theorem 2 is now complete, at least in the case where $A(t)$ is strictly increasing.

To complete the proof of Theorem 2 we drop the assumption that $A(t)$ be strictly increasing. It is still true however that $p(t) = N(A^{-1}(t))$ is a standard Poisson process with the property that $p(A(t)) = N(t)$ except possibly for an evanescent set and moreover matters can be arranged so that $A(t)$ is independent of $p(t)$ - see T. Brown (1981) Theorem 2 on

p. 308. Bring in the natural (strictly) increasing process $A_\varepsilon(t) = A(t) + \varepsilon t$ and note that $A_\varepsilon(t)$ decreases to $A(t)$ as ε decreases to 0 and therefore $\lim_{\varepsilon\to 0} p(A_\varepsilon(t)) = p(A(t))$ since p is right continuous - in particular $p(A_\varepsilon(t))$ converges weakly to $p(A(t))$. We observe that $N_\varepsilon(t) = p(A_\varepsilon(t))$ is again a counting process with strictly increasing compensator $A_\varepsilon(t)$. By Lemma 4 then

$$z_\varepsilon(t) = \exp(-\int_0^t f(u)dp(A_\varepsilon(u)) - \int_0^t [\exp(-f(u)) - 1]dA_\varepsilon(u))$$

is a martingale for every $\varepsilon > 0$. In order to pass to the limit as $\varepsilon \downarrow 0$ we first assume f is continuous and then use the weak convergence of $p(A_\varepsilon(u))$ to $p(A(u))$ to conclude

$$(2.17) \qquad \lim_{\varepsilon\to 0} \int_0^t f(u)dp(A_\varepsilon(u)) = \int_0^t f(u)dp(A(u))$$
$$= \int_0^t f(u)dN(u) \qquad \text{a.s.}$$

Similarly it is easy to check that

$$(2.18) \qquad \lim_{\varepsilon\to 0} \int_0^t [\exp(-f(u)) - 1]dA_\varepsilon(u) = \int_0^t [\exp(-f(u)) - 1]dA(u) \qquad \text{a.s.}$$

Clearly this implies that $\lim_{\varepsilon\to 0} z_\varepsilon(t) = z(t)$ is a martingale at least when $f(t)$ is continuous. Proceeding as we did just after (2.15) it can be shown that $z(t)$ is a martingale for step functions of the form (2.1) and finally the proof for arbitrary bounded predictable f is carried out by means of the standard approximation procedure used in (2.4)-(2.11). The proof of Theorem 2 is complete.

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